On possibility of realization of the phenomena of complex analytic dynamics in physical systems. Novel mechanism of the synchronization loss in coupled period-doubling systems

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Abstract

The possibility of realization of the phenomena of complex analytic dynamics for the realistic physical models are investigated. Observation of the Mandelbrot and Julia sets in the parameter and phase spaces both for the discrete maps and non-autonomous continuous systems is carried out. For these purposes, the method, based on consideration of coupled systems, demonstrating period-doubling cascade is suggested. Novel mechanism of synchronization loss in coupled systems corresponded to the dynamical behavior intrinsic to the complex analytic maps is offered.

1 Introduction

It is known [1, 2], that complex analytic dynamics (CAD), studying behavior of complex maps, includes a lot of interesting phenomena, for example, presence of fractal Mandelbrot and Julia sets in the parameter and phase spaces.

Let us start with a quadratic logistic map

$$z' \to \lambda - z^2, \tag{1}$$

where λ is a complex parameter, and z is a complex variable. By definition, Mandelbrot set (fig. 1) is a set of points on a plane of complex parameter λ, for which the orbit of an extremum $z = 0$ of the map (1) during iteration procedure does not escape to infinity. The Mandelbrot set contains the so-called "Mandelbrot cactus" (designed on figure by gray color); this is a set of points in the parameter plane, for which the trajectory starting of the extremum of the map converges to a periodic attractor.

"Mandelbrot cactus" consists of a big cardioid, corresponding to the existence of an attracting fixed point, and an infinite number of "leaves", corresponding to existence of attracting cycles of different periods. For example, the leaves of the doubled periods are placed along a real axis.

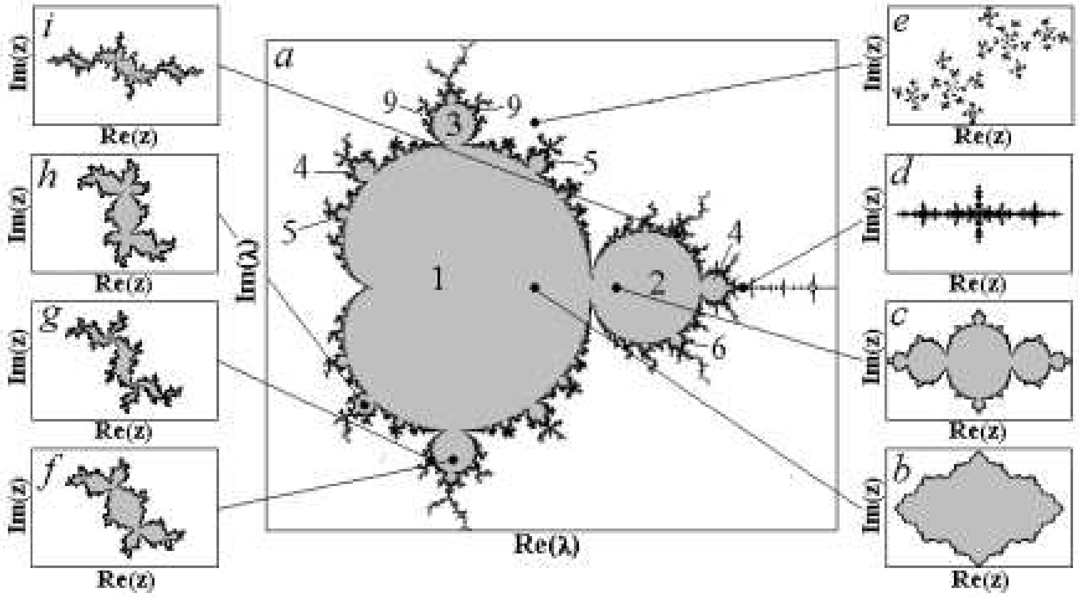

Figure 1: Mandelbrot set (a) and Julia sets for the quadratic complex map with different values of parameter: $\lambda = 0.5$ (b), $\lambda = 0.8$ (c), $\lambda = 1.42$ (d), $\lambda = 0.5 + 0.7i$ (e), $\lambda = 0.123 - 0.745i$ (f), $\lambda = 0.0315 - 0.7908i$ (g), $\lambda = -0.282 + 0.530i$ (h), $\lambda = 1.16 + 0.25i$ (i). The gray color designates the regions corresponding to existence of periodic dynamics (periods are marked by respective numbers); the black color designates points, at which the restricted in a phase space chaotic dynamics is implemented; the white color means the escaping of trajectories to infinity.

The sequences of other period m-tupling (period-multiplication) bifurcations also can be found. In particular, the accumulation points for period-tripling and period-quadrupling bifurcation cascades have been investigated in the work of Golberg, Sinai and Khanin [3]. In the works of Cvitanović and Mirheim [4, 5] the universal properties of many other bifurcation cascades were studied.

A bifurcation, which is responsible for originating a "leave", corresponds to a stability loss of a "parent" cycle, characterized by a complex multiplier with unit modulus and rational argument (in relation to 2π). If the argument of a multiplier at the stability loss is irrational, then the domains with fractal boundaries, filled by invariant curves arise in the phase plane, the so-called Siegel disks [6, 7, 8].

Fractal pattern close to the "Mandelbrot cactus" and denoted by black color in Fig. 1, corresponds to existence of the chaotic dynamical regimes in the phase space.

In figures 1(b-i) the Julia sets for different values of complex parameter λ are shown. The Julia set is a border between basins of attraction to infinity (white color) and to a periodic motion (gray color) in a plane of complex variable z. One can distinguish the following types of Julia sets:

- for values λ, belonging to the "Mandelbrot cactus", the Julia set is connected and enclose an interior basin (figs. 1(b,c,f-i);

- for values λ, at which chaotic dynamics exists, the Julia set is also connected, but has no inner region (fig. 1d);

- for values λ, outside the Mandelbrot set, the Julia set is disconnected (fig. 1f).

It is obvious, that 1D complex map can be represented equivalently by a 2D real map (for this purpose it is necessary only to separate real and imaginary parts of the equation). However, the mentioned phenomena of CAD are intrinsic only to a very special class of the real 2D maps, namely for the analytic maps, obeying the Cauchy-Riemann conditions. Violation of the analyticity leads to drastic changes of the dynamics of the map [9, 10, 11, 12, 13]. Thereby, a following problem arises: Is it possible to specify actual physical systems demonstrating phenomena of CAD? Recently, this problem attracts great attention. The physical applications of complex dynamics for such problems, as the renormalization group approach in the theory of phase transitions and the theory of a percolation were discussed [14, 15, 16, 17]. In the paper of Beck [18] a theoretical possibility of the construction of the physical system, in which the Mandelbrot set would arise, was considered. The suggested approach is based on analysis of a motion of a charged particle in a double-peak potential with non-linear damping. The particle is driven by magnetic field, depending on time and on the particle velocity, and effected by external shot pulses, time intervals between which also depend on the particle velocity.

In present work, we suggest a simpler and universal approach of constructing models manifesting the Mandelbrot set and other phenomena of CAD, which may be designed as realistic physical systems. This method allow us to carry out a physical experiment and present the first observation of the Mandelbrot set [19]. The special structure of the Fourier spectrum of signal generated by experimental system at the period-tripling accumulation point is presented in [20].

Our method is based on using of specially symmetrically coupled identical systems, demonstrating transition to chaos through period-doublings. As it is known, such behavior is peculiar for a very wide class of nonlinear dissipative systems of various physical nature. The special kind of offered coupling provides a special symmetry in the multi-dimensional system, which is necessary for implementation of the analyticity conditions. It is a simple problem to construct the system with such coupling in comparison to the system, suggested by Beck.

In section 2 the procedure of construction of the coupled systems demonstrating phenomena of CAD at the example of the discrete logistic maps is considered. In sections 3 and 4 we apply developed method to the various systems of Feigenbaum universality class, namely to the Hénon map and to the nonlinear non-autonomous oscillator. The relation of the CAD phenomena to the phenomena of generalized partial synchronization is discussed in section 5.

2 From complex quadratic map to coupled logistic maps

Let us start with the notion that one-dimensional complex quadratic map is equivalent to the system of two real coupled quadratic maps with a special type of coupling.

Separation of real and imaginary parts in the complex equation (1) yields

$$z'_{re} \to \lambda_{re} - z_{re}^2 + z_{im}^2, \quad z'_{im} \to \lambda_{im} - 2z_{re}z_{im}. \tag{2}$$

Next, we introduce the following designations

$$\begin{aligned} x_1 = z_{re} + \beta z_{im}, \quad x_2 = z_{re} - \beta z_{im}, \\ \lambda_1 = \lambda_{re} + \beta\lambda_{im}, \quad \lambda_2 = \lambda_{re} - \beta\lambda_{im}. \end{aligned} \tag{3}$$

As a result we obtain a system of two coupled quadratic maps

$$\begin{aligned} x_1' &\to \lambda_1 - x_1^2 + \varepsilon(x_2 - x_1)^2, \\ x_2' &\to \lambda_2 - x_2^2 + \varepsilon(x_1 - x_2)^2, \end{aligned} \tag{4}$$

where $\varepsilon = (1+\beta^2)/4\beta^2$ is the parameter of coupling. Note a special type of coupling in these equations. It can be interpreted as an identical simultaneous shift of control parameters in both partial systems, proportional to the squared difference of dynamic variables at each step of discrete time.

It is worth nothing that the coefficient $\varepsilon = (1+\beta^2)/4\beta^2$ for any β is larger than 1/4. Nevertheless, formally we can investigate system (4) with any ε.

In fig. 2 we present the charts of the parameter plane (λ_1, λ_2) for the coupled maps (4) at several values of parameter ε. One can see the usual Mandelbrot set, rotated by 45°, takes place at $\varepsilon = 0.5$. For $0.25 < \varepsilon < +\infty$ we have a distorted Mandelbrot set on the parameter plane. The cactus leaves of this Mandelbrot set correspond to existence of periodic motion of different periods. At $\varepsilon = 0.25$, the set on the parameter plane, for which the point starting from the origin does not escape to infinity, looks like a set of strips, where the period doubling cycles occur. At $\varepsilon < 0.25$ it transforms to a rhombus-like structure. At a particular $\varepsilon = 0$ (uncoupled logistic maps) it is a square.

The generalization of coupling to the case of $-\infty < \varepsilon < +\infty$ corresponds to the original map (1), variable and parameter of which are so-called two-component numbers [21, 22, 23, 24, 25, 26]. This is a special algebraic system, which elements are defined as follows

$$z = x + iy, \quad i^2 = a + ib, \quad \text{where} \quad a, b \in \mathbf{R}. \tag{5}$$

According to [21], there are three special cases: $i^2 = -1$ – the usual complex numbers, $i^2 = +1$ – the so-called perplex numbers, $i^2 = 0$ – the dual numbers. All other algebraic number systems are isomorphic to complex, perplex or dual numbers depending on, whether the value of $(a+b^2)/4b^2$ is positive, negative or zero, and are known as elliptic, hyperbolic or parabolic number system, respectively. In terms of parameter ε these conditions look as follows:

1) the case $\varepsilon > 0.25$ corresponds to elliptic numbers isomorphic to complex numbers, implemented at $\varepsilon = 0.5$;

2) the case $\varepsilon < 0.25$ corresponds to hyperbolic numbers isomorphic to perplex numbers, implemented at $\varepsilon = 0$;

3) the case $\varepsilon = 0.25$ corresponds to parabolic or dual numbers.

Thus, the existence of three topologically different structures on the plane of parameters (λ_1, λ_2), namely, fractal structure, similar to Mandelbrot set, rhombus-like structure and system of strips, is explained by existence of three different algebraic systems of numbers.

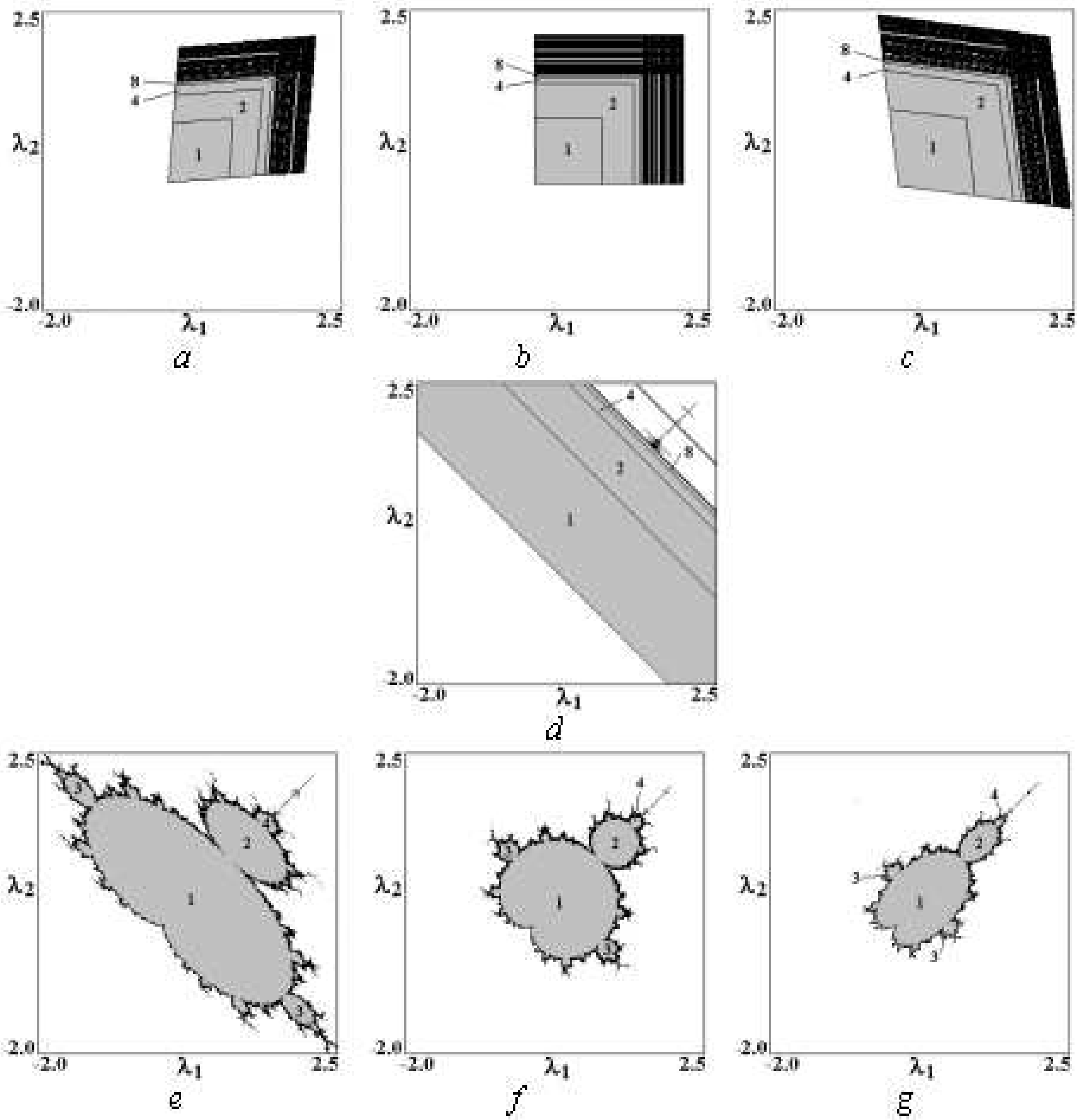

Figure 2: The charts of the parameter plane (λ_1, λ_2) for the coupled logistic maps (4) with different values of parameter of coupling: $\varepsilon = -0.1$ (a), $\varepsilon = 0.0$ (b), $\varepsilon = 0.1$ (c), $\varepsilon = 0.25$ (d), $\varepsilon = 0.3$ (e), $\varepsilon = 0.5$ (f), $\varepsilon = 1.0$ (g). The figures (a-c) correspond to hyperbolic numbers, figure (d) – to parabolic, and figures (e-g) – to elliptic numbers.

3 Coupled Hénon maps

The more realistic model, than the logistic map, is the Hénon map [27, 28]

$$\begin{aligned} x' &\to \lambda - x^2 - d \cdot y, \\ y' &\to x, \end{aligned} \tag{6}$$

which is two-dimensional reversible map and, therefore, can be realized as the Poincare cross-section of some continuous system with 3-D phase space – the minimal dimensionality ensuring an possibility of nontrivial dynamics and chaos.

In present section we show, that it is possible to observe the phenomena of CAD in the system of two coupled Hénon maps. (In [29] it is shown, that phenomena of CAD such as period multiplication cascades survives in complex Hénon map) For this purpose we carry out the following discourse, which starting point is the complexification of the individual map (6).

Let us consider dynamical variables x and y and driving parameter λ responsible to transition to chaos through period-doublings as complex, that is $x = x_{re} + ix_{im}$, $y = y_{re} + iy_{im}$, $\lambda = \lambda_{re} + i\lambda_{im}$. Parameter d we consider as real. Separating real and imaginary parts, we obtain:

$$\begin{aligned} x'_{re} &\to \lambda_{re} - x_{re}^2 + x_{im}^2 - d \cdot y_{re}, \\ x'_{im} &\to \lambda_{im} - 2x_{re}x_{im} - d \cdot y_{im}, \\ y'_{re} &\to x_{re}, \\ y'_{im} &\to x_{im}. \end{aligned} \tag{7}$$

Let us enter new variables and parameters

$$\begin{aligned} x_{1,2} &= x_{re} \pm \beta x_{im}, \\ y_{1,2} &= y_{re} \pm \beta y_{im}, \\ \lambda_{1,2} &= \lambda_{re} \pm \beta \lambda_{im}, \\ \varepsilon &= (1 + \beta^2)/4\beta^2. \end{aligned} \tag{8}$$

Then, the complexified Hénon map, represented in the form of coupled real maps looks as follows

$$\begin{aligned} x'_1 &\to \lambda_1 - x_1^2 - d \cdot y_1 + \varepsilon(x_2 - x_1)^2, \\ y'_1 &\to x_1, \\ x'_2 &\to \lambda_2 - x_2^2 - d \cdot y_2 + \varepsilon(x_1 - x_2)^2, \\ y'_2 &\to x_2. \end{aligned} \tag{9}$$

At fig. 3 the charts of the plane of parameters (λ_1, λ_2) for various values of parameter d are shown. As one can see, the parameter plane contains the domains similar by the shape to the Mandelbrot set, distorting with increasing of $|d|$. At the next figure the Julia sets for system (9) are represented. They also have apparent similarity to usual Julia sets. The basins of attraction of the fixed point (fig. 4a), cycles of period 2 (fig. 4b) and 3 (fig. 4c) corresponding to the filled-in Julia sets, and basin of attraction of chaotic attractor (fig. 4d) corresponding to the connected dendrit-like Julia set are shown. All figures in present and next sections are corresponded to the value of coupling parameter $\varepsilon = 0.5$ – the value, which is equivalent to usual complex numbers and usual Mandelbrot set.

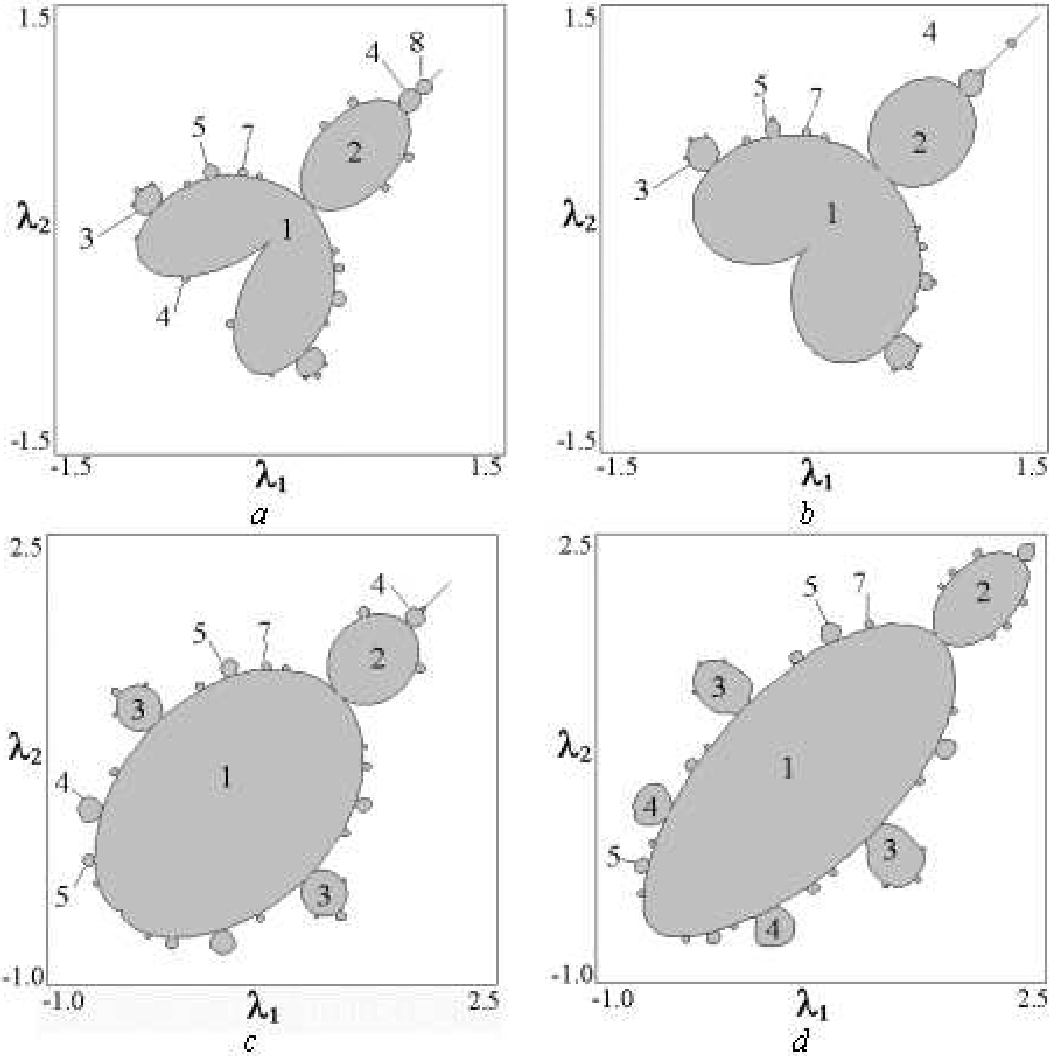

Figure 3: The Mandelbrot set for coupled Hénon mappings (9) with the following values of parameter d: -0.5 (a); -0.3 (b); 0.3 (c); 0.5 (d). Parameter of coupling $\varepsilon = 0.5$.

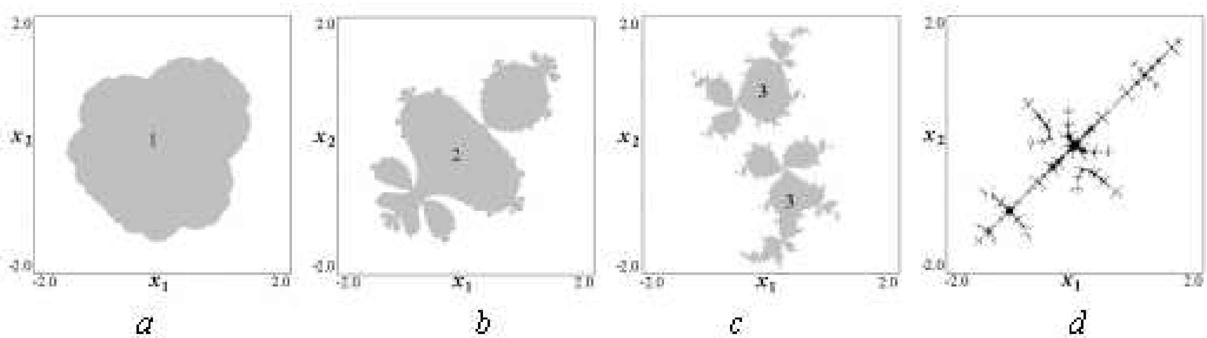

Figure 4: The Julia sets for coupled Hénon mappings with $d = -0.3$ for the following values of parameters λ_1 and λ_2: $\lambda_1 = \lambda_2 = 0.1$ (a); $\lambda_1 = \lambda_2 = 0.5$ (b); $\lambda_1 = -0.8, \lambda_2 = 0.5$ (c); $\lambda_1 = \lambda_2 = 1.05$ (d).

4 Coupled nonlinear oscillators

One of the most universal models, suitable for description of a number of real physical systems, is the nonlinear oscillator. Let us consider a quadratic oscillator with damping and harmonic driving force

$$\ddot{x} + \gamma\dot{x} + \lambda + x^2 = F\cos\omega t, \tag{10}$$

where x – dynamical variable,λ – parameter of nonlinearity, γ – parameter of damping, F and ω – amplitude and frequency of external driving force. It is known, that such systems demonstrate the opportunity of transition to chaos through a cascade of period-doubling bifurcations, for example by changing of parameter λ with fixed F, ω and γ [30, 31] (see fig. 5a).

Let us construct system of two coupled oscillators demonstrating phenomena of CAD, operating the same scheme as in the previous sections. At first, we complexify the equation of a quadratic oscillator in such way, that driving parameter λ and variable x are complex, and then, we make following designations

$$\begin{aligned} x_{1,2} &= x_{re} \pm \beta x_{im}, \\ \lambda_{1,2} &= \lambda_{re} \pm \beta\lambda_{im}, \\ \varepsilon &= (1+\beta^2)/4\beta^2. \end{aligned} \tag{11}$$

As a result we obtain the system of coupled oscillators

$$\begin{aligned} \ddot{x}_1 + \gamma\dot{x}_1 + \lambda_1 + x_1^2 - \varepsilon(x_2 - x_1)^2 &= F\cos\omega t, \\ \ddot{x}_2 + \gamma\dot{x}_2 + \lambda_2 + x_2^2 - \varepsilon(x_1 - x_2)^2 &= F\cos\omega t. \end{aligned} \tag{12}$$

At fig. 5b the chart of a plane of parameters (λ_1, λ_2) is exhibited. Although, the represented picture differ by the shape from the Mandelbrot set for coupled logistic maps, but it have its basic properties, such as presence of leaves of every possible periods and their self-similar organization. The difference is determined by the existence of the region of bistability for the quadratic oscillator (see fig. 5a).

Analogy between basins of periodic attractors for coupled oscillators represented at fig. 6 and usual Julia sets is also obvious. With $\lambda_1 = \lambda_2 = -0.8$ the basin of one attracting point is realized (figs. 6a). Then, it splits up to two basins of different fixed points (fig. 6b). With $\lambda_1 = \lambda_2 = -0.575$ (fig. 6c) one can see, that one of the basins destroys and with $\lambda_1 = \lambda_2 = -0.5$ there is only one attracting fixed point on the phase plane (fig. 6d). Figs. 6e-f demonstrate the basins of attracting cycles of period 2 and 3. It is necessary to note, that for non-autonomous coupled oscillators there can be a number of basins of attraction of various periodic and chaotic motion on the phase plane (x_1, x_2). Thus, only one of them, posed at a central part of figures is associated with Julia set corresponded to the Mandelbrot set of figure 5b.

Figure 7 represents the phase planes $(x_1, \dot{x}_1)$, $(x_2, \dot{x}_2)$ with the same values of parameters as for the fig. 6.

Thus, the strategy of construction of the flow system manifesting CAD phenomena is follow (see scheme at fig. 8): One should take two identical elements (describable by the continuous differential equations), make special coupling between them and synchronize them by common periodic external driving force.

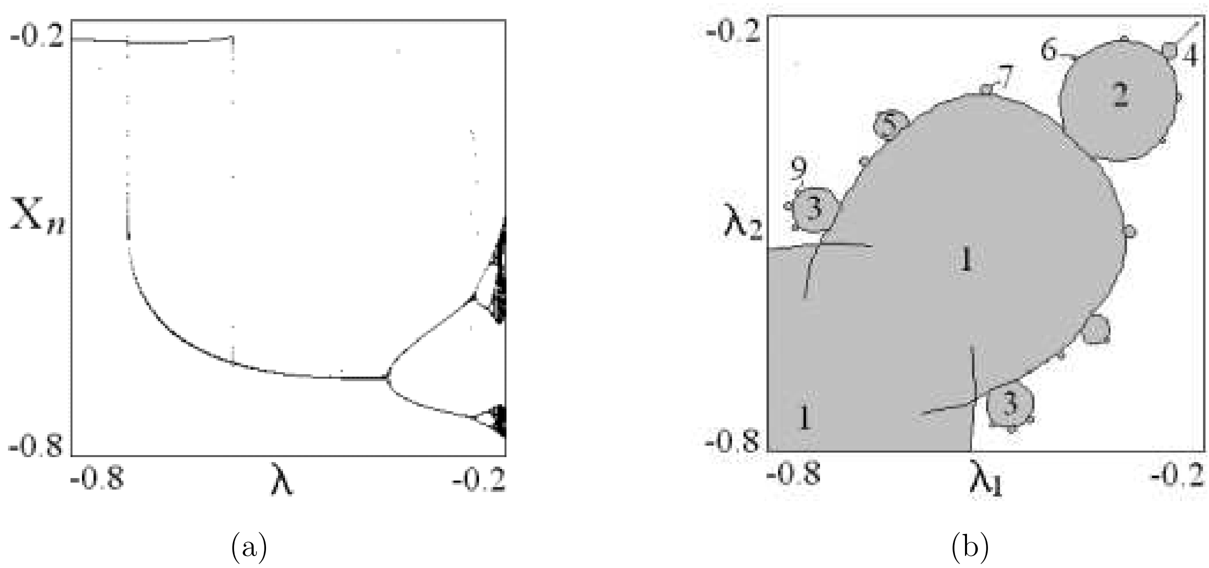

Figure 5: (a): Bifurcation tree for the quadratic oscillator (10) at the plane (X_n, λ), where X_n are the values of dynamical variable x at the Poincare cross-section $t = 2\pi n$ $(n = 1, 2, ...)$. Cascade of period-doubling bifurcations and region of multi-stability is visible. (b): The charts of dynamical regimes on a plane of parameters (λ_1, λ_2) for system of coupled oscillators (12) with $\gamma = 0.2$, $F = 0.23$, $\omega = 1$, $\varepsilon = 0.5$.

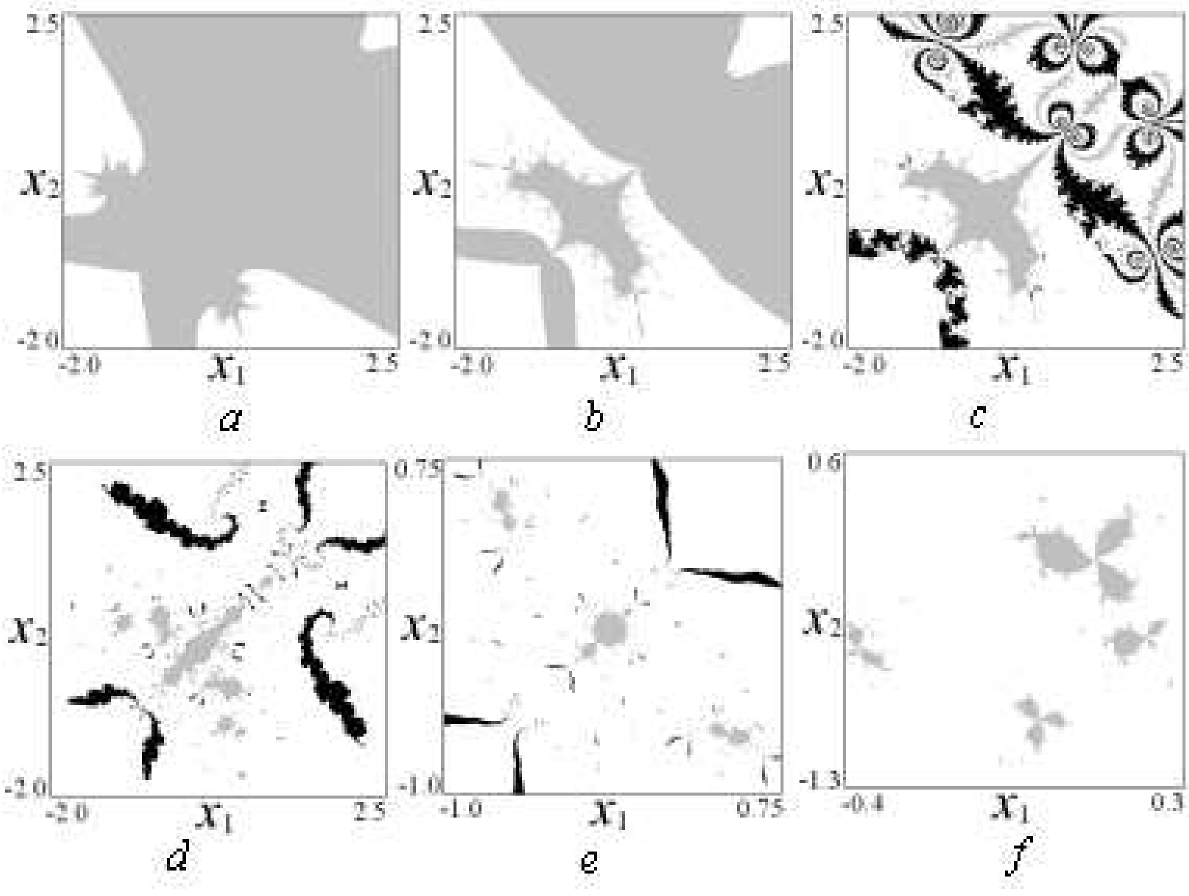

Figure 6: The charts of the phase space cross-section (x_1, x_2) for coupled quadratic oscillators with following values of parameters: $\lambda_1 = \lambda_2 = -0.8$ (a), $\lambda_1 = \lambda_2 = -0.6$ (b), $\lambda_1 = \lambda_2 = -0.575$ (c), $\lambda_1 = \lambda_2 = -0.5$ (d), $\lambda_1 = \lambda_2 = -0.3$ (e), $\lambda_1 = -0.74, \lambda_2 = -0.45$ (f). The basins of attraction of the fixed point (figs. a-d), and cycles of period 2 (fig. e) and 3 (fig. f) at the Poincare cross-section are represented.

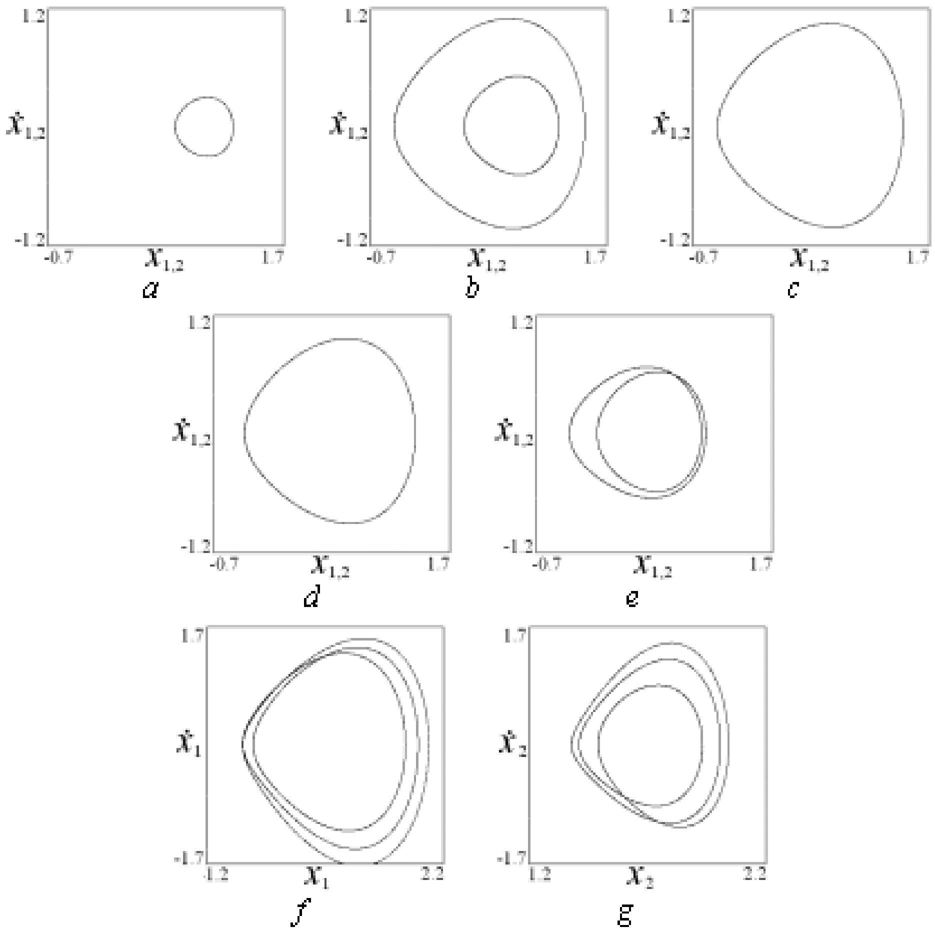

Figure 7: The charts of the phase plane $(x_1, \dot{x}_1)$ and $(x_2, \dot{x}_2)$ for coupled quadratic oscillators with following values of parameters: $\lambda_1 = \lambda_2 = -0.8$ (a), $\lambda_1 = \lambda_2 = -0.6$ (b), $\lambda_1 = \lambda_2 = -0.575$ (c), $\lambda_1 = \lambda_2 = -0.5$ (d), $\lambda_1 = \lambda_2 = -0.3$ (e), $\lambda_1 = -0.74, \lambda_2 = -0.45$ (f).

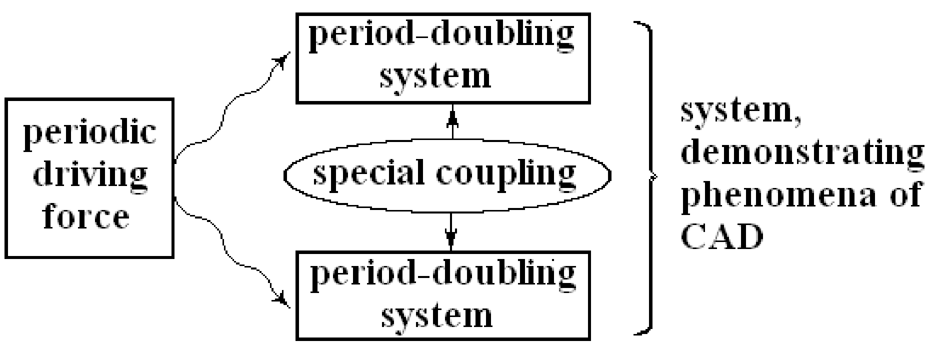

Figure 8: Scheme of construction of the physical system with CAD phenomena.

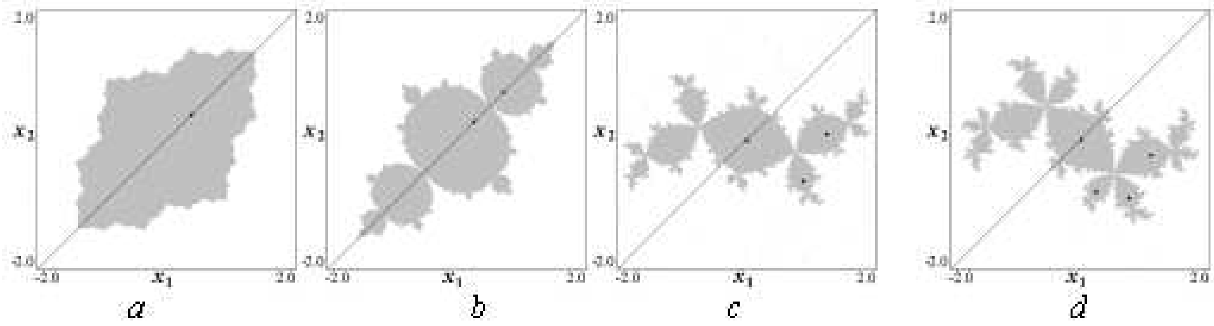

Figure 9: Charts of the phase plane (x_1, x_2) for the coupled logistic mappings with $\varepsilon = 0.5$ for the following values of parameters λ_1 and λ_2: $\lambda_1 = \lambda_2 = 0.5$ (a); $\lambda_1 = \lambda_2 = 0.8$ (b); $\lambda_1 = 0.868, \lambda_2 = 0.622$ (c); $\lambda_1 = 0.248, \lambda_2 = 0.812$ (d). The dots designate the attractors, which basins are marked by the gray color. The figures for a fixed point (a) and cycles of period 2 (b), 3 (c) and 4 (d) are represented.

5 Phenomena of CAD and mechanisms of synchronization loss of coupled systems

The investigation of phenomenon of synchronization loss in coupled systems demonstrating transition to chaos through period-doubling bifurcations is one of the mostly interesting problems of nonlinear dynamics and has a great fundamental and applied importance for different fields of science and technique. It is necessary to note, that the set of domains of periodic dynamics corresponded to leaves of Mandelbrot cactus, can be considered as the region of generalized partial synchronization. By generalized partial synchronization we mean the dynamical state of a system, in which the value $(x_1^2 + x_2^2)^{1/2}$ is bounded and varies periodically. It means, that the trajectories of one subsystem do not escape far from trajectories of other subsystem. At fig. 9 the charts of phase planes for coupled logistic maps with various values of λ_1 and λ_2 are represented. At these pictures, the bold dots indicate the periodic attractors, which basins of attraction are marked by the gray color. It is easy to see, that in the case $\lambda_1 = \lambda_2$ there is full synchronization in the system (attractor takes place at the diagonal line). In the case, when the parameters of subsystems λ_1 and λ_2 are not equal, but belongs to the "Mandelbrot cactus", the generalized synchronization of subsystems exists (periodic attractor takes place at the neighborhood of a diagonal line).

Let us demonstrate the idea of generalized partial synchronization also with an example of phase portraits of coupled oscillators. One can see, that with identical values of parameters λ_1 and λ_2 (see fig. 8a-e) the values of variables in subsystems are coincided, that corresponds to full synchronization. In a case $\lambda_1 \neq \lambda_2$ (fig. 8f,g), when the phenomena of CAD can be realized, the values of dynamical variables of subsystems do not coincide, that corresponds to realization of generalized partial synchronization. Thus, the phenomena of CAD such as period-tripling bifurcations can be implemented, when the point (λ_1, λ_2) belongs to region of generalized partial synchronization.

6 Conclusion

In present work we have offered a novel universal method for obtaining of the phenomena of CAD in the realistic models of physical systems. It is shown, that complexified system can be represented as two coupled real systems. Such representation is useful for some reasons. At first, often it simplify equations and construction of a real physical system. Secondly, with entering of special coupling to the system of two identical devices of any nature demonstrating the period-doublings cascade, one may expect the whole system to demonstrate phenomena of complex analytic dynamics.

The coupled logistic maps, coupled invertible Hénon maps and coupled nonlinear oscillators with periodic driving have been considered. Thus, we have studied both discrete maps, and non-autonomous system with continuous time.

It has been shown, that realization of the Mandelbrot set in the parameter space of the systems with discreet time and non-autonomous periodically driven systems is easy problem enough. It is necessary to consider two elements demonstrating period-doublings with coupling, arising from the complexification of the variables and parameter responsible for the period doublings in the original system. The special interest is attracted by the problem of realization of phenomena of CAD in the autonomous continuous systems.

Besides it is necessary to mark connection of the phenomena of CAD with a problem of synchronization. It is shown, that Mandelbrot set corresponds to the domain of generalized partial synchronization of coupled systems. Thus, the possibility of new nontrivial features of dynamics corresponded to synchronization of complex systems has been found.

Acknowledgements

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